Capture Region for True Proportional Navigation Guidance with Nonzero Miss-Distance

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Introduction

TRUE proportional navigation (TPN) is a version of proportional navigation (PN) guidance law in which the missile lateral acceleration is applied perpendicular (normal) to the line of sight (LOS)[1] rather than to the missile velocity, as in the case of pure proportional navigation (PPN).[2] The TPN guidance law has been found to be analytically more tractable than the PPN guidance law. Guelman[1] obtained the capture region of TPN, for a nonmaneuvering target, defining capture in terms of zero miss-distance. However, guided missiles carry warheads that have a nonzero lethal radius and thus, in practical situations, it is possible to define capture in terms of an acceptable nonzero miss-distance. It is obvious that the capture region, as obtained by Guelman,[1] for zero miss-distance will expand further under this assumption. In a previous paper,[3] the capture region for TPN with nonzero miss-distance was obtained through extensive computations. In this Note we present a complete analytical solution to the problem.

Problem Formulation and Preliminary Analysis

The planar geometry for missile-target engagement, when the missile follows a TPN guidance law, is shown in Fig. 1. The equations of motion are obtained as

$$V_r = \dot{r} = V_m \cos\alpha - V_t \cos\Theta \qquad (1)$$

$$V_\Theta = r\dot{\Theta} = V_m \sin\alpha + V_t \sin\Theta \qquad (2)$$

$$\dot{\tau} = -(a_m \cos\alpha / V_m) = \dot{\alpha} + \dot{\Theta} \qquad (3)$$

$$\dot{V}_m = -a_m \sin\alpha \qquad (4)$$

$$a_m = c\dot{\Theta} \qquad (5)$$

where $c>0$ is a constant and V_r and V_Θ are the relative velocity components along and normal to the LOS, respectively. The other variables are as shown in Fig. 1.

Differentiating Eqs. (1) and (2) with respect to time and performing some simple manipulations (as given in Ref. 1), we obtain the following capture equation:

$$r\ddot{r} + (\dot{r})^2 + 2c\dot{r} = a \qquad (6)$$

with

$$a = V_{r0}^2 + V_{\Theta 0}^2 + 2cV_{r0} \qquad (7)$$

where V_{r0} and $V_{\Theta 0}$ are the values of V_r and V_Θ at initial time $t = 0$. Letting

$$z = \frac{\mathrm{d}r}{\mathrm{d}t} \qquad (8)$$

we obtain

$$\ddot{r} = \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = z\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right) \qquad (9)$$

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which, when substituted in Eq. (6), yields

$$zr\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right) + z^2 + 2cz = a \qquad (10)$$

Separating the variables, we get

$$\frac{z}{a - z^2 - 2cz}\,\mathrm{d}z = \frac{\mathrm{d}r}{r} \qquad (11)$$

which can be rewritten as

$$\frac{z}{(a + c^2) - (z + c)^2}\,\mathrm{d}z = \frac{\mathrm{d}r}{r} \qquad (12)$$

Substituting

$$\mu^2 = a + c^2, \qquad y = z + c \qquad (13)$$

we get

$$\frac{y - c}{\mu^2 - y^2}\,\mathrm{d}y = \frac{\mathrm{d}r}{r} \qquad (14)$$

which can be integrated to yield

$$(\mu/c)\ln(\mu^2 - y^2) + \ln[(\mu + y)/(\mu - y)] + (2\mu/c)\ln(r) = k \qquad (15)$$

The constant of integration k can be obtained by substituting values at initial time $t = 0$. This substitution yields the following relation:

$$(\mu + y)^2(\mu^2 - y^2)^{\mu/c - 1} = (\mu + y_0)^2(\mu^2 - y_0^2)^{\mu/c - 1}(r_0/r)^{2\mu/c} \qquad (16)$$

where the subscript 0 represents the values at initial time $t = 0$. From Eqs. (1), (2), and (6) we can obtain

$$\mu^2 - y^2 = V_\Theta^2 \qquad (17)$$

Hence, Eq. (16) can be rewritten as

$$(\mu + y)^2\,(V_\Theta^2)^{\mu/c - 1} = (\mu + y_0)^2(V_{\Theta 0}^2)^{\mu/c - 1}(r_0/r)^{2\mu/c} \qquad (18)$$

This equation must hold at all points on the state trajectory. If $V_{\Theta 0} \neq 0$, then the right-hand side of Eq. (18) tends to ∞ for all values of μ and c as $r \to 0$. But the left-hand side of Eq. (18) can tend to infinity if $\mu < c$ and $V_\Theta \to 0$ as $r \to 0$. Note that if Eq. (18) is considered in isolation then the left-hand side $\to \infty$ if $y \to \infty$, or if $\mu > c$ and $V_\Theta \to \infty$. However, either of these conditions is impossible since Eq. (17) has to be satisfied all along the trajectory. That reasoning yields the capture condition for zero miss-distance, with $V_{\Theta 0} \neq 0$, as

$$\mu < c \qquad (19)$$

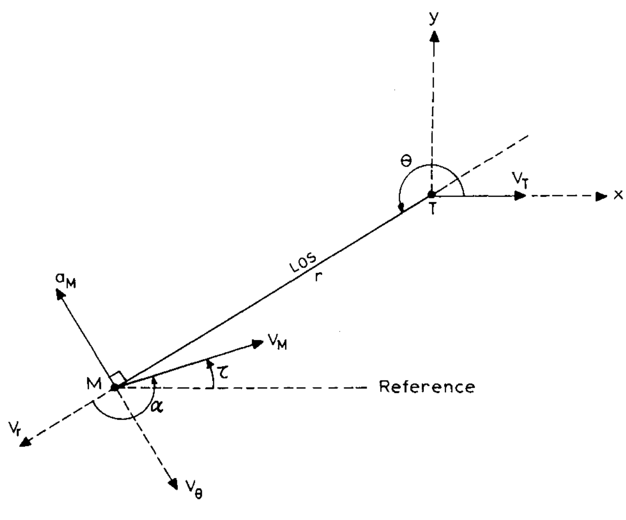

Fig. 1 Missile-target engagement geometry.

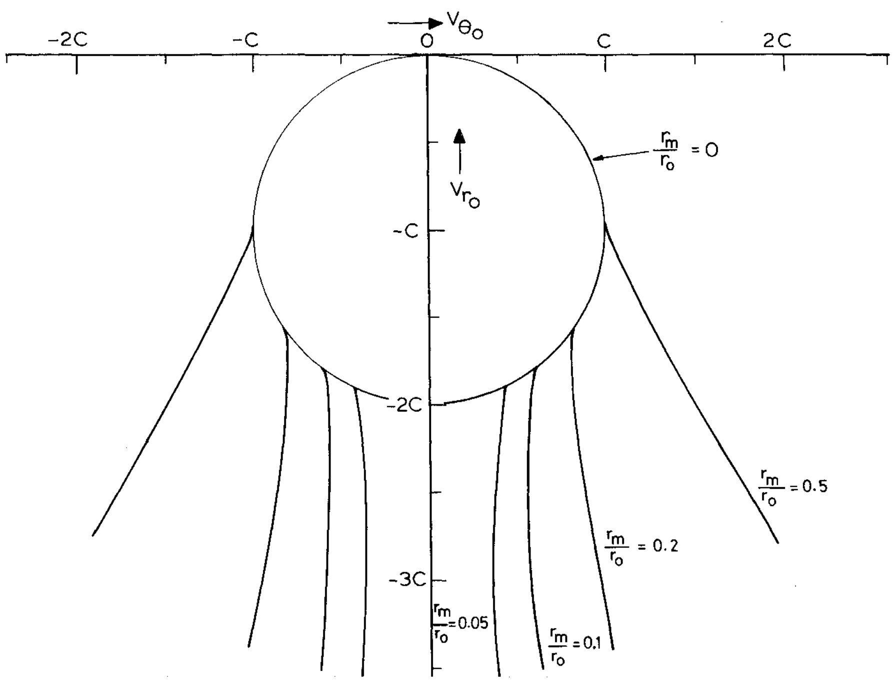

Fig. 2 Capture regions for nonzero miss-distance.

which leads to the inequality

$$a < 0 \tag{20}$$

which is the capture condition for zero miss-distance. This was the condition obtained by Guelman in Ref. 1. This condition represents the interior of a circle given by the equation

$$(V_{r0} + c)^2 + V_{\Theta 0}^2 = c^2 \tag{21}$$

in the $(V_{\Theta 0}, V_{r0})$ space of initial conditions.

Capture Region for Nonzero Miss-Distance

Let $r_m > 0$ be the acceptable miss-distance for capture to occur. Since we have already proved that capture occurs with zero miss-distance only when $\mu < c$, we need consider only those points in the initial condition space for which $\mu \geq c$.

Now, consider an engagement in which the miss-distance is exactly r_m. At the instant when miss-distance occurs (i.e., the instant of closest approach), we get

$$V_r = 0 \tag{22}$$

Thus, at that instant

$$y = c \tag{23}$$

Substituting in Eq. (16) we obtain

$$(\mu + c)^2(\mu^2 - c^2)^{\mu/c - 1} = (\mu + y_0)^2(V_{\Theta 0}^2)^{\mu/c - 1}(r_0/r_m)^{2\mu/c} \tag{24}$$

This expression can be further simplified to yield

$$r_m = r_0(\alpha_1)^{(1 + c/\mu)/2}(\alpha_2)^{(1 - c/\mu)/2} \tag{25}$$

where

$$\alpha_1 = 1 + V_{r0}/(\mu + c) \tag{26}$$

$$\alpha_2 = 1 - V_{r0}/(\mu - c) \tag{27}$$

Hence, given V_{r0} and $V_{\Theta 0}$, Eq. (25) gives the expected miss-distance. Thus, if the acceptable miss-distance is specified to be $r_m > 0$ then the capture region in the $(V_{\Theta 0}, V_{r0})$ space is given by those points $(V_{\Theta 0}, V_{r0})$, $V_{r0} < 0$, which satisfy the inequality

$$(\alpha_1)^{(1 + c/\mu)/2}(\alpha_2)^{(1 - c/\mu)/2} \leq (r_m/r_0) \tag{28}$$

Note that this inequality is valid for $\mu > c$ and the region defined by this inequality contains the capture region for zero miss-distance. Now, Eq. (28) can be used to obtain the capture region for various values of (r_m/r_0).

A geometric interpretation of Eq. (28), which allows the capture region to be obtained easily, is as follows: μ is the Euclidean distance of the initial point $(V_{\Theta 0}, V_{r0})$ from the center of the zero miss-distance capture circle, i.e., $(0, -c)$. Given a value of $r_m/r_0 < 1$, we can draw a circle N with radius $\mu > c$ and center at $(0, -c)$. Now we find the value of $V_{r0} = p$ (say), in the region $V_{r0} < 0$, such that Eq. (28) is satisfied with the equality sign. The intersection of the circle N and the line $V_{r0} = p$ gives a point on the boundary of the capture region, on both sides of the V_{r0} axis, corresponding to the given value of r_m/r_0. In Fig. 2, this procedure is adopted to obtain capture regions for various values of r_m/r_0. It is observed that the capture region remains almost unchanged for low values of initial closing velocity ($= -V_{r0}$), but expands significantly for high values of closing velocity.

Conclusions

In this Note, an analytical solution to the problem of obtaining the capture region of a missile pursuing a nonmaneuvering target and using a TPN guidance law, with acceptable nonzero miss-distance, is presented. It is shown that the capture region is enlarged, and deviates from its circular shape, in comparison with the zero miss-distance case. In contrast to previous results, the capture region is obtained without solving the equations of motion in closed form.

References

[1]Guelman, M., "The Closed-Form Solution of True Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-12, No. 4, 1976, pp. 472–482.

[2]Guelman, M., "A Qualitative Study of Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-7, No. 4, 1972, pp. 637–643.

[3]Dhar, A., and Ghose, D., "Capture Region for a Realistic True Proportional Navigation Guidance Law," *IEEE Transactions on Aerospace and Electronic Systems*, AES-29, No. 3, 1993, pp. 995–1003.